A stability version for a theorem of Erdős on nonhamiltonian graphs

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Abstract

Let n, d be integers with $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$, and set $h(n,d) := \binom{n-d}{2} + d^2$ and $e(n,d) := \max\{h(n,d), h(n, \lfloor \frac{n-1}{2} \rfloor)\}$. Because $h(n,d)$ is quadratic in d, there exists a $d_0(n) = (n/6) + O(1)$ such that

$$e(n,1) > e(n,2) > \cdots > e(n,d_0) = e(n,d_0+1) = \cdots = e(n, \left\lfloor \frac{n-1}{2} \right\rfloor).$$

A theorem by Erdős states that for $d \le \lfloor \frac{n-1}{2} \rfloor$, any n-vertex nonhamiltonian graph G with minimum degree $\delta(G) \ge d$ has at most $e(n,d)$ edges, and for $d \ge d_0(n)$ the unique sharpness example is simply the graph $K_n - E(K_{\lceil (n+1)/2 \rceil})$. Erdős also presented a sharpness example $H_{n,d}$ for each $1 \le d \le d_0(n)$.

We show that if $d < d_0(n)$ and a 2-connected, nonhamiltonian n-vertex graph G with $\delta(G) \ge d$ has more than $e(n, d+1)$ edges, then G is a subgraph of $H_{n,d}$. Note that $e(n,d) - e(n,d+1) = n - 3d - 2 \ge n/2$ whenever $d < d_0(n) - 1$.

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Dedicated to the memory of Professor H. Sachs.

1 Introduction

We use standard notation. In particular, $V(G)$ denotes the vertex set of a graph G, $E(G)$ denotes the edge set of G, and $e(G) = |E(G)|$. Also, if $v \in V(G)$, then $N(v)$ denotes the neighborhood of v and $d(v) = |N(v)|$. Ore [4] proved the following Turán-type result:

Theorem 1 (Ore [4]). *If G is a nonhamiltonian graph on n vertices, then $e(G) \le \binom{n-1}{2} + 1$.*

This bound is achieved only for the n-vertex graph obtained from the complete graph K_{n-1} by adding a vertex of degree 1. Erdős [2] refined the bound in terms of the minimum degree of the graph:

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Theorem 2 (Erdős [2])**.** *Let n, d be integers with $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$, and set $h(n,d) := \binom{n-d}{2} + d^2$. If G is a nonhamiltonian graph on n vertices with minimum degree $\delta(G) \ge d$, then*

$$e(G) \le \max\left\{h(n,d), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\} =: e(n,d).$$

This bound is sharp for all $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$.

To show the sharpness of the bound, for $n, d \in \mathbb{N}$ with $d \le \lfloor \frac{n-1}{2} \rfloor$, consider the graph $H_{n,d}$ obtained from a copy of K_{n-d}, say with vertex set A, by adding d vertices of degree d each of which is adjacent to the same d vertices in A. An example of $H_{11,3}$ is below.

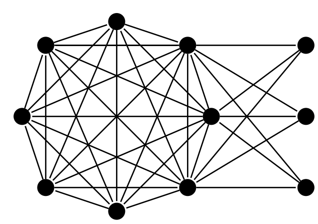

Figure 1: $H_{11,3}$

By construction, $H_{n,d}$ has minimum degree d, is nonhamiltonian, and $e(H_{n,d}) = \binom{n-d}{2} + d^2 = h(n,d)$. Elementary calculation shows that $h(n,d) > h(n, \lfloor \frac{n-1}{2} \rfloor)$ in the range $1 \le d \le \lfloor \frac{n-1}{2} \rfloor$ if and only if $d < (n+1)/6$ and n is odd or $d < (n+4)/6$ and n is even. Hence there exists a $d_0 := d_0(n)$ such that

$$e(n,1) > e(n,2) > \cdots > e(n,d_0) = e(n,d_0+1) = \cdots = e(n, \left\lfloor \frac{n-1}{2} \right\rfloor),$$

where $d_0(n) := \lceil \frac{n+1}{6} \rceil$ if n is odd, and $d_0(n) := \lceil \frac{n+4}{6} \rceil$ if n is even. Let $H'_{n,d}$ denote the graph that is an edge-disjoint union of two complete graphs K_{n-d} and K_{d+1} sharing one vertex.

The result of this note is the following refinement of Theorem 2.

Theorem 3. *Let $n \ge 3$ and $d \le \lfloor \frac{n-1}{2} \rfloor$. Suppose that G is an n-vertex nonhamiltonian graph with minimum degree $\delta(G) \ge d$ such that*

$$e(G) > e(n,d+1) = \max\left\{h(n,d+1), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\right\}. \tag{1}$$

(So we have $d < d_0(n)$.) Then G is a subgraph of either $H_{n,d}$ or $H'_{n,d}$.

This is a stability result in the sense that for $d < n/6$, each 2-connected, nonhamilitonian n-vertex graph with minimum degree at least d and "close" to $h(n,d)$ edges is a subgraph of the extremal graph $H_{n,d}$. Note that $h(n,d) - h(n,d+1) = n - 3d - 2$ is at least $n/2$ for $d < d_0 - 1$. Note also that $e(H'_{n,d}) > e(n,d+1)$ only when $d = O(\sqrt{n})$.

We will use the following well-known theorems of Pósa.

Theorem 4 (Pósa [5])**.** *Let $n \ge 3$. If G is a nonhamiltonian n-vertex graph, then there exists $1 \le k \le \lfloor \frac{n-1}{2} \rfloor$ such that G has a set of k vertices with degree at most k.*

Theorem 5 (Pósa [6]). *Let $n \geq 3$, $1 \leq \ell < n$ and let G be an n-vertex graph such that $d(u) + d(v) \geq n + \ell$ for every non-edge uv in G. Then for every linear forest F with ℓ edges contained in G, the graph G has a hamiltonian cycle containing all edges of F.*

2 Proof of Theorem 3

Call a graph G *saturated* if G is nonhamiltonian but for each $uv \notin E(G)$, $G + uv$ has a hamiltonian cycle. Ore's proof [4] of Dirac's Theorem [1] yields that

$$\text{for every } n\text{-vertex saturated graph } G \text{ and for each } uv \notin E(G),\ d(u) + d(v) \leq n - 1. \qquad (2)$$

First we show two facts on saturated graphs with many edges.

Lemma 6. *Let G be a saturated n-vertex graph with $e(G) > h(n, \lfloor \frac{n-1}{2} \rfloor)$. Then for some $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, $V(G)$ contains a subset D of k vertices of degree at most k such that $G - D$ is a complete graph.*

Proof. Since G is nonhamiltonian, by Theorem 4, there exists some $1 \leq k \leq \lfloor \frac{n-1}{2} \rfloor$ such that G has k vertices with degree at most k. Pick the maximum such k, and let D be the set of the vertices with degree at most k. Since $e(G) > h(n, \lfloor \frac{n-1}{2} \rfloor)$, $k < \lfloor \frac{n-1}{2} \rfloor$. So, by the maximality of k, $|D| = k$.

Suppose there exist $x, y \in V(G) - D$ such that $xy \notin E(G)$. Among all such pairs, choose x and y with the maximum $d(x)$. Since $y \notin D$, $d(y) > k$. Let $D' := V(G) - N(x) - \{x\}$ and $k' := |D'| = n - 1 - d(x)$. By (2),

$$d(z) \leq n - 1 - d(x) = k' \text{ for all } z \in D'. \qquad (3)$$

So D' is a set of k' vertices of degree at most k'. Since $y \in D'$, $k' \geq d(y) > k$. Thus by the maximality of k, we get $k' = n - 1 - d(x) > \lfloor \frac{n-1}{2} \rfloor$. Equivalently, $d(x) < \lceil \frac{n-1}{2} \rceil$. For all $z \in D' + \{x\}$, either $z \in D$ where $d(z) \leq k \leq \lfloor \frac{n-1}{2} \rfloor$, or $z \in V(G) - D$, and so $d(z) \leq d(x) \leq \lfloor \frac{n-1}{2} \rfloor$. It follows that $e(G) \leq h(n, \lfloor \frac{n-1}{2} \rfloor)$, a contradiction. □

Lemma 7. *Under the conditions of Lemma 6, if $k = \delta(G)$, then $G = H_{n,\delta(G)}$ or $G = H'_{n,\delta(G)}$.*

Proof. Set $d := \delta(G)$, and let D be a set of d vertices with degree at most d. Let $u \in D$. Since $\delta(G) \geq |D| = d$, u has a neighbor $w \in V(G) - D$. Consider any $v \in D - \{u\}$. By Lemma 6, w is adjacent to all of $V(G) - D - \{w\}$. It also is adjacent to u, therefore its degree is at least $n - d$. We obtain

$$d(w) + d(v) \geq (n - d) + d = n.$$

Then by (2), w is adjacent to v, and hence w is adjacent to all vertices of D.

Let W be the set of vertices in $V(G) - D$ having a neighbor in D. We have obtained that $W \neq \emptyset$ and

$$N(u) \cap (V(G) - D) = W \text{ for all } u \in D. \qquad (4)$$

Let $G' = G[D \cup W]$. If $|W| = 1$, then $G = H'_{n,d}$. If $|V(G')| = 2d$, then by (4), each vertex $u \in D$ has the same d neighbors in $V(G) - D$. Because $d(u) = d$, D is an independent set. Thus $G = H_{n,d}$. Otherwise, $d + 2 \leq |V(G')| \leq 2d - 1$, $|D| \geq 2$.

Fix a pair of vertices $w_1, w_2 \in W$. For any $x, y \in V(G')$,

$$d(x) + d(y) \geq d + d \geq |V(G')| + 1.$$

Therefore by Theorem 5, G' has a hamiltonian cycle C that uses the edge w_1w_2. Since $G'' := G - (V(G') - \{w_1, w_2\})$ is a complete graph, it contains a hamiltonian w_1, w_2-path P. Then $P \cup (C - w_1w_2)$ is a hamiltonian cycle of G, a contradiction. □

Proof of Theorem 3. Suppose that an n-vertex, nonhamiltonian graph G satisfies the constraints of Theorem 3 for some $1 \leq d \leq \left\lfloor \frac{n-1}{2} \right\rfloor$. We may assume G is saturated, since if a graph containing G is a subgraph of $H_{n,d}$ or $H'_{n,d}$, then G is as well.

By Lemma 6, G has a set D of $k \leq \left\lfloor \frac{n-1}{2} \right\rfloor$ vertices with degree at most k such that $G - D$ is a complete graph. Therefore $e(G) \leq \binom{n-k}{2} + k^2 = h(n,k)$. If $k > d$, then $e(G) \leq \max\{h(n, d+1), h(n, \left\lfloor \frac{n-1}{2} \right\rfloor)\} = e(n, d+1)$, a contradiction. Thus $k \leq d$. Furthermore, $k \geq \delta(G) \geq d$, and hence $k = d$. Also, since $e(G) > h(n, \left\lfloor \frac{n-1}{2} \right\rfloor))$, we have $d + 1 \leq d_0(n) \leq (n+8)/6$. Applying Lemma 7 completes the proof. □

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